Complex Analysis (for Physics)

Midterm Exam (with solutions)

Problem 1

Prove that if $|z| = 1$ then $\left|\dfrac{\bar{b}z + \overline{a}}{az + b}\right| = 1$ for all complex numbers a, b, $(a, b) \neq (0, 0)$.

Solution 1.1

Using the multiplicative property of the modulus, i.e. $|z_1 z_2| = |z_1||z_2|$, and $|\overline{z}| = |z| = 1$ and $z\overline{z} = |z|^2 = 1$ we have

$$\left|\frac{\bar{b}z + \overline{a}}{az + b}\right| \overset{(1)}{=} |\overline{z}| \left|\frac{\bar{b}z + \overline{a}}{az + b}\right| \overset{(2)}{=} \left|\frac{\bar{b}z\overline{z} + \overline{a}\,\overline{z}}{az + b}\right| \overset{(3)}{=} \left|\frac{\bar{b} + \overline{a}\,\overline{z}}{az + b}\right| \overset{(4)}{=} \left|\frac{\overline{b + az}}{az + b}\right| \overset{(5)}{=} \frac{|\overline{b + az}|}{|az + b|} \overset{(6)}{=} \frac{|b + az|}{|az + b|} = 1.$$

Above we used (1) $|\overline{z}| = |z| = 1$, (2) $|z_1|\,|z_2| = |z_1 z_2|$, (3) $z\overline{z} = |z|^2 = 1$, (4) $\overline{z}_1\,\overline{z}_2 = \overline{z_1 z_2}$ and $\overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}$, (5) $|z_1/z_2| = |z_1|/|z_2|$, (6) $|\overline{z}_1| = |z_1|$

Solution 1.2

We can check if we have $|\bar{b}z + \overline{a}| = |az + b|$.
We start with $|\bar{b}z + \overline{a}|^2$.

$$\begin{aligned}
|\bar{b}z + \overline{a}|^2 &= (\bar{b}z + \overline{a})\overline{(\bar{b}z + \overline{a})} \\
&= (\bar{b}z + \overline{a})(b\overline{z} + a) \\
&= b\bar{b}z\overline{z} + \bar{b}za + b\overline{z}\overline{a} + a\overline{a} \\
&= |bz|^2 + |a|^2 + \bar{b}za + b\overline{z}\overline{a} \\
&= |b|^2|z|^2 + |a|^2 + \bar{b}za + b\overline{z}\overline{a}.
\end{aligned}$$

Similarly, we can look at $|az + b|^2$.

$$\begin{aligned}
|az + b|^2 &= (az + b)\overline{(az + b)} \\
&= (az + b)(\overline{a}\overline{z} + \bar{b}) \\
&= a\overline{a}z\overline{z} + az\bar{b} + \overline{a}\overline{z}b + b\bar{b} \\
&= |az|^2 + |b|^2 + az\bar{b} + \overline{a}\overline{z}b \\
&= |a|^2|z|^2 + |b|^2 + az\bar{b} + \overline{a}\overline{z}b.
\end{aligned}$$

Note that if $|z| = 1$, then $|z|^2 = 1$. This means that if $|z| = 1$, then $|\bar{b}z + \overline{a}|^2 = |az + b|^2$.
Since moduli are always non-negative, we have $|\bar{b}z + \overline{a}| = |az + b|$.
Note that we should have $a \neq -b/z$ in order for the modulus of $|az + b|$ to not be zero. This excludes a few more points than just $(a, b) = (0, 0)$.

Solution 1.3

An other way to solve this, is by using $z = x + yi$ with $x, y \in \mathbb{R}$, and writing $a = c + di$, $b = e + fi$, with $c, d, e, f \in \mathbb{R}$. With this you can use $|z|^2 = x^2 + y^2$. All the steps are equivalent to steps above.

Problem 2

Find all complex number solutions of the equation $z^2 + |z| = 0$. Write your final answer in algebraic form.

Solution 2.1

Writing z in exponential form, i.e. $z = re^{i\theta}$ with $r = |z| \geq 0$ and $\theta = \mathrm{Arg}(z) \in [-\pi, \pi)$, turns the equation $z^2 + |z|$ into $e^{2i\theta}r^2 + r = 0$. The left-hand side can be factored as

$$r(e^{2i\theta}r + 1) = 0 \iff r = 0 \quad \text{or} \quad re^{2i\theta} = -1.$$

In the former case, we have $z = 0e^{i\theta} = 0$ whereas the latter equation is satisfied if and only if $r = 1$ and $2\theta = \pi k$ for $k \in \mathbb{Z}$, which means $\theta = \pi/2k$, which gives two solutions $z = \pm i$. Hence the equation $z^2 + |z| = 0$ has three complex number solutions: $z \in \{0, -i, i\}$.

Solution 2.2

Rewriting the equation as $z^2 = -|z|$ and taking the modulus of both sides result in

$$|z^2| = |-|z|| = |z| \quad \iff \quad |z^2| - |z| = 0 \quad \iff \quad |z|(|z| - 1) = 0.$$

Thus $|z| = 0$ (i.e. $z = 0$) or $|z| = 1$ (i.e. z lies on the unit circle). Either way $|z^2| = |z|$ implies that the equation can be written as $z^2 + |z|^2 = 0$ which may be rewritten as $z(z + \overline{z}) = 0$, which is the case when $z = 0$ or $2\mathrm{Re}(z) = z + \overline{z} = 0$, that is the case if and only if z is purely imaginary. The only purely imaginary numbers on the unit circle are $\pm i$. Hence the equation $z^2 + |z| = 0$ has three complex number solutions: $z \in \{0, -i, i\}$.

Problem 3

Show that the complex function $w = z + \dfrac{1}{z}$ maps the circles $|z| = r$ (with $r > 1$) onto ellipses. What happens when $r \to 1$?

Solution

Fix $r > 1$ and take $z \in \mathbb{C}$ such that $|z| = r$, i.e. $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ for some $\theta \in [-\pi, \pi)$. Then $z^{-1} = r^{-1}(\cos\theta - i\sin\theta)$ and w can be written as follows

$$w = z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

Introducing the notation $a := r + r^{-1}$, $b := r - r^{-1}$ and taking the real and imaginary parts of the equation above, we get $\mathrm{Re}\, w = a\cos\theta$ and $\mathrm{Im}\, w = b\sin\theta$. Therefore

$$\frac{(\mathrm{Re}\, w)^2}{a^2} + \frac{(\mathrm{Im}\, w)^2}{b^2} = \cos^2\theta + \sin^2\theta = 1,$$

which is the equation of the ellipse centred at the origin with a and b as its semi-major and semi-minor axes along the x- and y-axis, respectively. Thus w lies on an ellipse. It's also clear that different angles $\theta \in [-\pi, \pi)$ result in different pairs $(\cos\theta, \sin\theta)$, so w as a mapping from the xy-plane to the uv-plane gives a one-to-one correspondence between the circle $x^2 + y^2 = r^2$ and the ellipse $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$.

We see that $\lim\limits_{r\to 1^+} a = \lim\limits_{r\to 1^+}(r + r^{-1}) = 1 + 1 = 2$ and $\lim\limits_{r\to 1^+} b = \lim\limits_{r\to 1^+}(r - r^{-1}) = 1 - 1 = 0$, i.e. the semi-major axis tends to 2 and the semi-minor axis vanishes. Thus in the limit $r \to 1$, the ellipse becomes the line segment connecting $(-2, 0)$ and $(2, 0)$ in the uv-plane.

Problem 4

Consider the complex function $f(x+iy) = (x^2+2y) + i(x^2+y^2)$ and determine the points $z_0 \in \mathbb{C}$ at which the derivative $f'(z_0)$ exists.

Solution

Since the function f is defined and continuously differentiable everywhere in the xy-plane, therefore f is complex differentiable at $z_0 = x_0 + iy_0$ if and only if the Cauchy-Riemann equations hold at z_0. By computing the first partial derivatives, we get

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2, \quad \frac{\partial v}{\partial y} = 2y.$$

Hence the Cauchy-Riemann equations read

$$\begin{cases} \dfrac{\partial u}{\partial x} &= \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} &= -\dfrac{\partial v}{\partial x} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 2x &= 2y \\ 2 &= -2x \end{cases}$$

The second condition implies that $x = -1$, which when plugged into the first equation yields $y = -1$. Therefore $z_0 = -1 - i$ is the *only* point where $f'(z_0)$ exists, $f'(-1-i) = -2(1+i)$.

Problem 5

Determine the points at which the complex function $g(z) = \dfrac{1}{(1-\sin z)^2}$ has no derivative and compute its derivative where it exists.

Solution

First, we find where the function is not differentiable. The function $g(z) = \dfrac{1}{(1-\sin z)^2}$ is an elementary function, because it can be obtained using a finite number of basic operations $+, -, \times, \div, \circ, ()^{-1}$ and the functions i, z, e^z. Therefore it is differentiable at every point where it can be defined (i.e. on its natural domain). Consequently, it has no derivative at the points where it is undefined. These are the points where the denominator vanishes, that is when $(1-\sin z)^2 = 0$, i.e. $\sin z = 1$. Solving this equation for z (using for example the exponential definition of sine) we obtain $z = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$. Therefore $g(z)$ is not differentiable at $z = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$.

Everywhere else we may use the Quotient Rule, and the Chain Rule (or the Power Rule and Chain Rule) to compute the derivative of $g(z)$. We obtain

$$g'(z) = \frac{2\cos z}{(1-\sin z)^3}.$$

Problem 6

Verify that the function $v(x,y) = y + e^{x^2-y^2}\sin 2xy$ is harmonic in $\mathbb{C}$ and find a harmonic conjugate $-u(x,y)$ such that $u(0,0) = 3$.

Solution 6.1

Note that $v(x, y)$ is defined and continuously differentiable (to arbitrary order) everywhere. Furthermore, note that if $z = x + iy$, then

$$e^{z^2} = e^{(x^2+y^2)+i(2xy)} = e^{x^2-y^2} e^{i(2xy)} = e^{x^2-y^2}(\cos 2xy + i \sin 2xy).$$

Therefore

$$\operatorname{Im}(z + e^{z^2}) = y + e^{x^2-y^2} \sin 2xy = v(x, y).$$

So $v(x, y)$ is harmonic in $\mathbb{C}$, because it is the imaginary part of the entire function $z + e^{z^2}$. The real part of this function is also harmonic everywhere and takes the following form

$$\operatorname{Re}(z + e^{z^2}) = x + e^{x^2-y^2} \cos 2xy.$$

Since this function assumes the value 1 at $z = 0$, but we want the real part to evaluate to 3 at $z = 0$, we consider the entire function $f(z) = 2 + z + e^{z^2}$ instead. This leaves the imaginary part unchanged, i.e. $\operatorname{Im} f(z) = v(x, y)$. The real part $u(x, y) = \operatorname{Re} f(z) = 2 + x + e^{x^2-y^2} \cos 2xy$ is still harmonic everywhere, $-u(x, y)$ serves as a harmonic conjugate to $v(x, y)$ and $u(0, 0) = 2 + 0 + e^0 \cos 0 = 3$, indeed.

Solution 6.2

Note that $v(x, y)$ is defined and continuously differentiable (to arbitrary order) everywhere. To verify that $v(x, y)$ is harmonic in $\mathbb{C}$, it remains to show that it satisfies Laplace's equation for any $x, y \in \mathbb{R}$. To compute the partial derivatives we will make use of the Product Rule, the Chain Rule and Basic Derivatives. The first partial derivatives are

$$\begin{aligned}
\frac{\partial v}{\partial x} &= 2xe^{x^2-y^2} \sin(2xy) + 2ye^{x^2-y^2} \cos(2xy), \\
\frac{\partial v}{\partial y} &= 1 - 2ye^{x^2-y^2} \sin(2xy) + 2xe^{x^2-y^2} \cos(2xy).
\end{aligned}$$

The pure second partial derivatives are

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} =& 2e^{x^2-y^2} \sin(2xy) + 4x^2 e^{x^2-y^2} \sin(2xy) + 4xye^{x^2-y^2} \cos(2xy) \\
&+ 4xye^{x^2-y^2} \cos(2xy) - 4y^2 e^{x^2-y^2} \sin(2xy), \\
\frac{\partial^2 v}{\partial y^2} =& -2e^{x^2-y^2} \sin(2xy) + 4y^2 e^{x^2-y^2} \sin(2xy) - 4xye^{x^2-y^2} \cos(2xy) \\
&- 4xye^{x^2-y^2} \cos(2xy) - 4xye^{x^2-y^2} \sin(2xy).
\end{aligned}$$

Clearly, the second order partial derivatives are continuous and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \quad \text{for any } x, y \in \mathbb{R}.$$

Therefore $v(x, y)$ is harmonic in the whole plane. To find its harmonic conjugate we solve the Cauchy-Riemann equations for $u(x, y)$. Integrating $-\frac{\partial v}{\partial x}$ with respect to y, we get

$$u(x, y) = -\int (2xe^{x^2-y^2} \sin(2xy) + 2ye^{x^2-y^2} \cos(2xy))\, dy = e^{x^2-y^2} \cos(2xy) + h(x).$$

Let us now differentiate $u(x, y)$ with respect to x using the Product Rule, Chain Rule, and Basic Derivatives:

$$\frac{\partial u}{\partial u} = -2xe^{x^2-y^2}\cos(2xy) + 2ye^{x^2-y^2}\sin(2xy) + h'(x).$$

Thus to satisfy the Cauchy-Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, we need to have

$$2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) + h'(x) = 1 - 2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy),$$

i.e. $h'(x) = 1$, implying that $h(x) = x + C$, where C is an arbitrary real constant. Thus

$$u(x, y) = e^{x^2-y^2}\cos(2xy) + x + C.$$

By applying the initial condition we can find the value of C,

$$u(0, 0) = e^0\cos(0) + 0 + C = 3 \Rightarrow C = 2.$$

Hence, the complex conjugate $-u(x, y)$ is given by

$$u(x, y) = e^{x^2-y^2}\cos(2xy) + x + 2.$$